Collected Papers

Bernhard Riemann

Translated from the 1892 edition by Roger Baker, Charles Christenson
and Henry Orde

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Preface

Bernhard Riemann (1826–1866) was one of the greatest mathematicians of the modern era. He aimed very deliberately at new ways of thinking about existing problems and concepts in mathematics, often with startling success. Looking for a quantitative way to support the assertion that he influenced twentieth century mathematics more than his contemporaries, I turned to the collection 'Development of Mathematics 1900–1950', edited by J-P. Pier. Riemann is mentioned in the index as many times as Gauss, Cauchy, Weierstrass and Dedekind combined.

Over the years, I have often talked to mathematicians and students who would like to read Riemann's papers, but find this difficult or time-consuming in the original German. This is the audience I had in mind in organizing the present volume. A few of the papers have been translated before, but the translations in this book are all new.

Riemann was fortunate enough to attend the lectures (on the least squares method) of the legendary C.F. Gauss (1777–1855) as a young student in Göttingen. He came to idolize Gauss later, but in 1847 he traveled to Berlin for greater stimulus. There he attended the lectures, and took part in the discussions, of C.G.J. Jacobi (1804–1851) and P.G.L. Dirichlet (1805–1859). Both these scholars became important influences on Riemann. He cited Jacobi's great memoir 'Fundamenta nova theoriae functionum ellipticarum' in many places in his own work, and it suggested possibilities to him that went well beyond Jacobi's beautiful calculations. Dirichlet, on the other hand, liked to take an abstract approach to each topic, and Riemann much preferred this way of attacking problems. Dirichlet was also an enthusiastic proponent of mathematical models of physical problems, and Riemann was to carry Dirichlet's approach further in both his writings and his lectures. Returning to Göttingen, he also became fired with enthusiasm for experimental physics and the construction of theoretical explanations for the new phenomena that were being observed. Here his great mentor was Wilhelm Weber (1804–1891).

As a young researcher, Riemann was drawn in many directions. However, his doctoral thesis of 1851 on the foundations of the theory of functions of a complex variable took on great importance in suggesting to him several further lines of research that needed to be carried through. The thesis also became a watershed in the subject, and in 1951 was celebrated with a centennial conference. There were many inspired ideas in the thesis. Riemann surfaces appear there for the first time. Analytic functions are viewed as conformal mappings, and the Riemann mapping theorem is proposed and given a proof, albeit unsatisfactory. Riemann's researches later in the 1850s on hypergeometric functions, Abelian functions and the Riemann zeta function demonstrate his commitment to complex function theory as a tool of exploration. Yet Dieudonné (1985) writes of Riemann's great paper **VI** on Abelian functions as an 'epoch' in algebraic geometry. It took a very long time before all the mathematical ideas in this paper fully bore fruit. Even a brilliant contemporary such as A. Clebsch (1833–1872) felt somewhat defeated by Riemann's memoir, although Clebsch took up the potent idea of genus of an algebraic curve from it. The paper **VII** on the Riemann zeta function, treasured by number theorists as a gem, is the source of the most famous unsolved problem in mathematics, the 'Riemann hypothesis'.

Simply in preparing for his habilitation in 1854, Riemann wrote two works of genius—his habilitation thesis on trigonometric series, and his 'trial lecture' on the foundations of geometry. The methods in the thesis can still be found intact in modern works such as Zygmund's 'Trigonometric Series'. The trial lecture became the inspiration for a new era in differential geometry, and Einstein's theory of general relativity is its (not very indirect) descendant. Yet Riemann never published either the thesis or the trial lecture—this was left to his friend and colleague Richard Dedekind (1831–1916) after Riemann's death. In 1854, these works simply qualified Riemann to be a poorly paid instructor at Göttingen.

It is hard to believe, yet while working on these wonderful ideas, Riemann still had a great deal of time to think about problems of physics. Of the nine papers that he published during his lifetime, four are on problems of mathematical physics. (Riemann's total of publications up to 1866 is brought up to eleven by announcements of his papers on the hypergeometric functions and the propagation of sound waves.) Much of his writing on physics from the 1850s was not submitted for publication.

It is difficult to guess the directions in which Riemann might later have focused his extraordinary abilities. He succeeded Dirichlet as professor at Göttingen in 1859, but his health deteriorated from 1862 onwards. He remained dedicated to scholarship to the last; this touching story is told here in an essay by Dedekind. Among his contemporaries, Dedekind was the one who most fully appreciated Riemann's mathematics.

After Riemann died in Italy in 1866, Dedekind and others oversaw the publication of seven posthumous papers. The early death of Clebsch delayed the appearance of the collected works, but eventually Heinrich Weber (1842–1913) was able to send the first edition to the press in 1876. Weber was ably assisted by Dedekind and by H.A. Schwarz (1848–1921). A dozen papers in the first edition were assembled by poring over Riemann's *Nachlass*, the mass of materials left behind at his death. The 1892 edition, which is the source for the present translations, contained some modest additions and corrections, and its numbering **I–XXXI** of the papers is preserved here for the reader's convenience. However, I omitted three non-mathematical items: **XVIII**, 'The mechanism of the ear', and two fragments on philosophy that can be found on pages 509–525 of Weber (1892).

Footnotes in the text of the papers are mostly Riemann's. However, some footnotes making Riemann's references to the literature more precise were added by Weber. These are indicated by a W.

There is far more in Riemann's work than any one person can comprehend if the influence of the papers is to be properly considered. Nevertheless, I provide notes at the end of the book containing basic information about each paper and suggestions for further reading.

A natural next step would be the book of Laugwitz (1909), which gives a unified account of Riemann as a mathematician and natural philosopher. Riemann also had a very important role as teacher and expositor through the revision and publication of his lectures by Hattendorff, Stahl and others.

Readers who would like to suggest corrections or alternative readings within the papers, or additional remarks for the notes, should send these to baker@math.byu.edu. I will maintain a web page that takes these suggestions into account, at http://www.math.byu.edu/~baker/Riemann/index.htm

During the final stages of the preparation of this book, Henry Orde, one of my fellow translators, died in Kent, England. Henry was born in 1922. He served in the British armed forces throughout the second world war, and then studied mathematics at Cambridge. After working for a firm of merchants in Malaya for several years, he joined the nascent computer industry in England. A heart attack forced him into retirement at age 50, and he was then able to devote his energies to pure mathematics, book collecting and music. Number theorists will recall that he gave an elementary proof of the class number formula for quadratic fields with negative discriminant. See Orde (1978).

Roger Baker

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